A NOTE ON MODIFIED UNBIASED RATIO-TYPE ESTIMATOR

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1. Introduction

In view of the fact that under simple random sampling the usual ratio estimator $\bar{y}/\bar{x}$, where $\bar{y}$ and $\bar{x}$ are sample means of y and x, is biased, one line to get an unbiased estimator, utilizing the auxiliary information, is to modify the sampling procedure so that the same estimator becomes unbiased. The other is to obtain the ratio-type estimator which is unbiased under simple random sampling. Hartley and Ross [2] corrected the estimator

$$\bar{r}=n^{-1}\sum^{n} y_i/x_i$$

for its bias and obtained the unbiased estimator of $R=Y/X$, where Y and X are the population totals for the variables y and x, as

$$\hat{R}=\bar{r}+\frac{n(N-1)}{(n-1)N}\quad\frac{(\bar{y}-\bar{r}\bar{x})}{\bar{X}} \qquad \text{...(1)}$$

where

$$\bar{X}=N^{-1}X \text{ (assumed known)},\ \bar{y}=n^{-1}\sum^{n} y_i$$

$$\bar{x}=n^{-1}\sum^{n} x_i,\ r_i=y_i/x_i,\ \bar{r}=n^{-1}\sum^{n} r_i$$

For this estimator when some of the values of the auxiliary variable x are zero, $\bar{r}$ and hence $\hat{R}$ does not exist. For such cases Shah and Adhvaryu [3] have modified the estimator by defining a

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new set of variables

$$x_i' = x_i + KX\ ; \text{ where } K>0 \text{ if } X>0 \text{ and } K<0 \text{ if } X<0,$$

$$y_i' = y_i \ (i=1, 2, \ldots N \text{ and } NK+1 \neq 0)$$

The unbiased estimator of R, using this set of variables is

$$t = (NK+1)\hat{R}', \qquad \ldots(2)$$

where

$$\hat{R}' = \bar{r}' + \frac{n(N-1)}{(n-1)N} \ \frac{(\bar{y}' - \bar{r}'\bar{x}')}{\bar{X}'};$$

$$\bar{r}' = \sum^{n} r_i'/n, \ ; \ r_i' = y_i'/x_i'$$

$$\bar{x}' = \sum_{i=1}^{N} x_i'/N$$

The variance of the estimator t is also given in [3]. In this note, the unbiased estimator of the variance of t is obtained in section 2 and section 3 is concerned with selection of K. Notations are same as in [3].

2. UNBIASED ESTIMATOR OF VARIANCE OF t

In the following Theorem 2·1 we obtain an unbiased estimator of the variance of t.

Theorem 2·1. The unbiased estimator of variance of t is given by

$$\hat{V}(t) = (NK+1)^2 \left[\frac{s^2_{r'}}{n} + \frac{2C_1'}{\bar{X}'(n-2)} + \frac{(n-1)\ s^2_{r'}\ s^2_x + (n-3)\ C_1^2 + \left(1-\frac{2}{n}\right)(n-1)\ k'_{22}}{\bar{X}^2(n^2-n-2)} \right] \qquad \ldots(3)$$

where

$$s^2_{r'} = (n-1)^{-1} \sum^{n} (r_i' - \bar{r}')^2, \quad s^2_x = (n-1)^{-1} \sum^{n} (x_i - \bar{x})^2 \qquad \ldots(4)$$

Computational formulas for C_1, C_1' and k'_{22}

(Fisher's k statistic) are given by

$$n(n-1)\, C_1 = n \sum^{n} y_1 - \left(\sum^{n} x'_i \right) \sum^{n} r'_i \,)$$

$$(n-1)\, C'_1 = \sum^{n} y_i r'_i - 2 \sum y_i + \bar{r}'^2 \sum^{n} x'_i - (n-1)\bar{X}' S^2_r\,,$$

$$k'_{22} = \frac{n}{(n-1)(n-2)(n-3)} \Big[(n+1) S'_{22} - \frac{2(n+1)}{n} S'_{21} S'_{01}$$

$$- \frac{2(n+1)}{n} S'_{12} S'_{10} - \frac{(n-1)}{n} S'_{20} S'_{02} - \frac{2(n-1)}{n} S'^2_{11}$$

$$+ \frac{8}{n} S'_{11} S'_{01} S'_{01} + \frac{2}{n} S'_{20} S'^2_{01} + \frac{2}{n} S'_{02} S'^2_{10} - \frac{6}{n^2} S'^2_{10} S'^2_{01} \Big] \quad \ldots(5)$$

where

$$S'_{tj} = \sum^{n} x'^t_i \, r'^j_i$$

i.e. $$S'_{22} = S_{22} = \sum^{n} y^2_i\,,\ S'_{21} = \sum^{n} y_i x'_i$$

$$S'_{12} = \sum^{n} y_i r'_i\,,\ S'_{20} = \sum^{n} x'^2_i\,,\ S'_{02} = \sum^{n} r'^2_i\,,$$

$$S'_{11} = S_{11} = \sum^{n} y_i,\ S'_{10} = \sum^{n} x'_i\,,\ S'_{01} = \sum^{n} r'_i$$

Proof : The unbiased estimator of the variance of $\hat{R}$ as given by Goodman and Hartley [1] is

$$\hat{V}(\hat{R}) = \frac{s^2_r}{n} + \frac{2C'}{\bar{X}(n-2)}$$

$$+ \frac{(n-1)\, s^2_r\, s^2_x + (n-3)\, C^2 + \left(1 - \frac{2}{n}\right)(n-1) k_{22}}{\bar{X}^2 (n^2 - n - 2)} \quad \ldots(6)$$

where s^2_r, s^2_x and k_{22} have the same meaning as in (4) and (5) omitting the dashes. While C and C' have the same meaning as C_1 and C'_1 omitting the dashes in their definitions. By using (6) we get the unbiased estimator of $V(\hat{R}')$ as

$$\hat{V}(\hat{R}')=\frac{s^2_{\cdot}}{n}+\frac{2C_{\cdot}'}{\bar{X}'(n-2)}$$

$$+\frac{(n-1)\,s^2_{r'}\,s^2_x+(n-3)C_1^2+\left(1-\frac{2}{n}\right)(n-1)\,k'_{22}}{\bar{X}'^2(n^2-n-2)} \quad \text{.. (7)}$$

Multiplying (7) by $(NK+1)^2$, we get the unbiased estimator of $V(t)$.

3. Selection of K

We select K such that the variance of the estimator t is minimized. This problem is solved approximately. The variance of t is

$$V(t)=\frac{1}{n\bar{X}^2}\left[S_y^2+\bar{R}'^2S_x^2-2\bar{R}'S_{xy}\right]$$

$$+\frac{1}{n(n-1)\bar{X}^2}\left[S^2_{r'}\,S_x^2+S^2_{r'x}\right]$$

(Result 2 of [3] with correction of a misprint)

where S_y^2, S_x^2 and S_{xy} have usual meanings and

$$S^2_{r'}=\sum_{i=1}^{N}(r_i'-\bar{R}')^2/N-1,$$

$$S_{r'x}=\sum_{i=1}^{N}(r_i'-\bar{R}')(x_i-\bar{x})/N-1, \bar{R}'=\sum_{i=1}^{N}r_i'/N$$

We omit the second term in this variance expression and minimise $V(t)$ with respect to K. Then we get

$$\sum_{i=1}^{N}\frac{y_i}{(x_i+KX)^2}=0$$

i.e.
$$\sum_{i=1}^{N}y_i\left(1+\frac{x_i}{KX}\right)^{-2}=0$$

Assuming x_i/KX to be small and ignoring higher power of x_i/KX

than one, we get

$$K=\frac{2\sum_{i=1}^{N} y_i x_i}{XY}$$

The term $2\sum^{n} y_i x_i/X \sum^{n} y_i$ can be used as an estimate of K.

REFERENCES

[1] Goodman, L.A. and Hartley, H.O. : "The precision of unbiased ratio-type estimator". *Journal of American Statistical Association*, 53 (1958), 491-508.

[2] Hartley, H.O. and Ross, A. : "Unbiased ratio estimators". *Nature*, 174 (1954), 270-271.

[3] Shah, S.M. and Adhvaryu, D. : "A modified unbiased ratio-type estimator". *Calcutta Statistical Association Bulletin*, 19 (1970), 159-161.